Formalization of DFA using lattices

Getting help

Are you getting enough help and support?

A. Yes

B. No because I didn’t realize there were office hours

C. No because the office hours are at a time that I can’t make

D. No because I’m embarrassed to ask for help

E. No because of some other reason

Project

How is the project going?

A. Easy, getting it all done quickly and easily

B. Challenging but doable

C. Very challenging, I’m having a hard time

D. Have no clue where to start

Recall worklist algorithm

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
   m(e) := ∅

for each node n do
   worklist.add(n)

while (worklist.empty.not) do
   let n := worklist.remove_any;
   let info_in := m(n.incoming_edges);
   let info_out := F(n, info_in);
   for i := 0 .. info_out.length do
      let new_info := m(n.outgoing_edges[i]) ∪
                      info_out[i];
      if (m(n.outgoing_edges[i]) ≠ new_info])
         m(n.outgoing_edges[i]) := new_info;
         worklist.add(n.outgoing_edges[i].dst);
```

Using lattices

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- But more generally ANY lattice
- What should be top and what should be bottom?

Using lattices

- We formalize our domain with a powerset lattice
- But more generally ANY lattice
- What should be top and what should be bottom?
- Does it matter?
  - It matters because, as we've seen, there is a notion of approximation, and this notion shows up in the lattice

Using lattices

- Unfortunately:
  - dataflow analysis community has picked one direction
  - abstract interpretation community has picked the other
- We will work with the abstract interpretation direction
- Bottom is the most precise (optimistic) answer, Top the most imprecise (conservative)

Direction of lattice

- Always safe to go up in the lattice
- Can always set the result to $\top$
- Hard to go down in the lattice
- Bottom will be the empty set in reaching defs

Worklist algorithm using lattices

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
   m(e) := ⊥

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                            info_out[i];
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- For reaching definitions, it terminates...
- Why?
  - lattice is finite
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  - Yes, we only require the lattice to have a finite height
- Height of a lattice: length of the longest ascending or descending chain
- Height of lattice (2^S, $\subseteq$) = ??

A. | S | - 1
B. | S |
C. | S | + 1
D. None of the above

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- Why?
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- Height of a lattice: length of the longest ascending or descending chain
- Height of lattice $(2^S, \subseteq) = | S |$

Termination

- Still, it's annoying to have to perform a join in the worklist algorithm

```
while (worklist.empty.not) do
   let n := worklist.remove_any;
   let info_in := m(n.incoming_edges);
   let info_out := F(n, info_in);
   for i := 0 .. info_out.length do
      let new_info := m(n.outgoing_edges[i]) ⊔
                      info_out[i];
      if (m(n.outgoing_edges[i]) ≠ new_info])
         m(n.outgoing_edges[i]) := new_info;
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```

- It would be nice to get rid of it, if there is a property of the flow functions that would allow us to do so

Even more formal

- To reason more formally about termination and precision, we re-express our worklist algorithm mathematically

- We will use fixed points to formalize our algorithm

Fixed points

- Recall, we are computing m, a map from edges to dataflow information
- Define a global flow function F as follows: F takes a map m as a parameter and returns a new map m', in which individual local flow functions have been applied

Fixed points

- We want to find a fixed point of F, that is to say a map m such that m = F(m)
- Approach to doing this?
- Define $\tilde{\perp}$, which is $\perp$ lifted to be a map:

  $\tilde{\perp} = \lambda$ e. $\perp$
- Compute $F(\tilde{\perp})$, then $F(F(\tilde{\perp}))$, then $F(F(F(\tilde{\perp})))$, ... until the result doesn't change anymore

Fixed points

- Formally:

$$Soln = \bigsqcup_{i=0}^{\infty} F^i(\tilde{\perp})$$

- Outer join has same role here as in worklist algorithm: guarantee that results keep increasing
- BUT: if the sequence $F^i(\tilde{\perp})$ for i = 0, 1, 2 ... is increasing, we can get rid of the outer join!

Fixed points

- Formally:

$$\mathrm{Soln} = \bigsqcup_{i=0}^{\infty} F^i(\tilde{\bot})$$

- Outer join has same role here as in worklist algorithm: guarantee that results keep increasing
- BUT: if the sequence $F^i(\tilde{\bot})$ for i = 0, 1, 2 ... is increasing, we can get rid of the outer join!
- How? Require that F be monotonic:
  - $\forall\, a, b\ .\ a \sqsubseteq b \Rightarrow F(a) \sqsubseteq F(b)$

Little bit more about monotonicity

- Definition: F is monotonic if and only if:
  - $\forall$ a, b . a $\sqsubseteq$ b $\Rightarrow$ F(a) $\sqsubseteq$ F(b)

- Which of the following is true:

A. If F is monotonic then $\forall$ a . F(a) $\sqsubseteq$ a

B. If F is monotonic then $\forall$ a . a $\sqsubseteq$ F(a)

C. If $\forall$ a . F(a) $\sqsubseteq$ a then F is monotonic

D. If $\forall$ a . a $\sqsubseteq$ F(a) then F is monotonic

E. None of the above

Fixed points

Fixed points

$$\tilde{I} \sqsubseteq F(\tilde{I})$$

$$F(\tilde{I}) \sqsubseteq F(F(\tilde{I}))$$

$$F^{k}(\tilde{I}) \sqsubseteq F^{k+1}(\tilde{I})$$

$$F^{k+1}(\tilde{I}) \sqsubseteq F^{k+2}(\tilde{I})$$

Back to termination

- So if F is monotonic, we have what we want: finite height $\Rightarrow$ termination, without the outer join
- Also, if the local flow functions are monotonic, then global flow function F is monotonic

Another benefit of monotonicity

- Suppose Marsians came to earth, and miraculously give you a fixed point of F, call it fp.
- Then:

Another benefit of monotonicity

- Suppose Marsians came to earth, and miraculously give you a fixed point of F, call it fp.
- Then:

$$\tilde{\bot} \sqsubseteq fp$$

$$F(\tilde{\bot}) \sqsubseteq F(fp)$$

$$F(\tilde{\bot}) \sqsubseteq fp$$

$$F^2(\tilde{\bot}) \sqsubseteq fp$$

$$\vdots$$

$$ofp \sqsubseteq fp$$

Another benefit of monotonicity

- We are computing the least fixed point...

Recap

- Let’s do a recap of what we’ve seen so far

- Started with worklist algorithm for reaching definitions

Worklist algorithm for reaching defns

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
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while (worklist.empty.not) do
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         m(n.outgoing_edges[i]) := new_info;
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```

Generalized algorithm using lattices

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
   m(e) := ⊥

for each node n do
   worklist.add(n)

while (worklist.empty.not) do
   let n := worklist.remove_any;
   let info_in := m(n.incoming_edges);
   let info_out := F(n, info_in);
   for i := 0 .. info_out.length do
      let new_info := m(n.outgoing_edges[i]) ⊔
                          info_out[i];
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         m(n.outgoing_edges[i]) := new_info;
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```

Next step: removed outer join

- Wanted to remove the outer join, while still providing termination guarantee

- To do this, we re-expressed our algorithm more formally

- We first defined a “global” flow function F, and then expressed our algorithm as a fixed point computation

Guarantees

- If F is monotonic, don’t need outer join
- If F is monotonic and height of lattice is finite: iterative algorithm terminates
- If F is monotonic, the fixed point we find is the least fixed point.

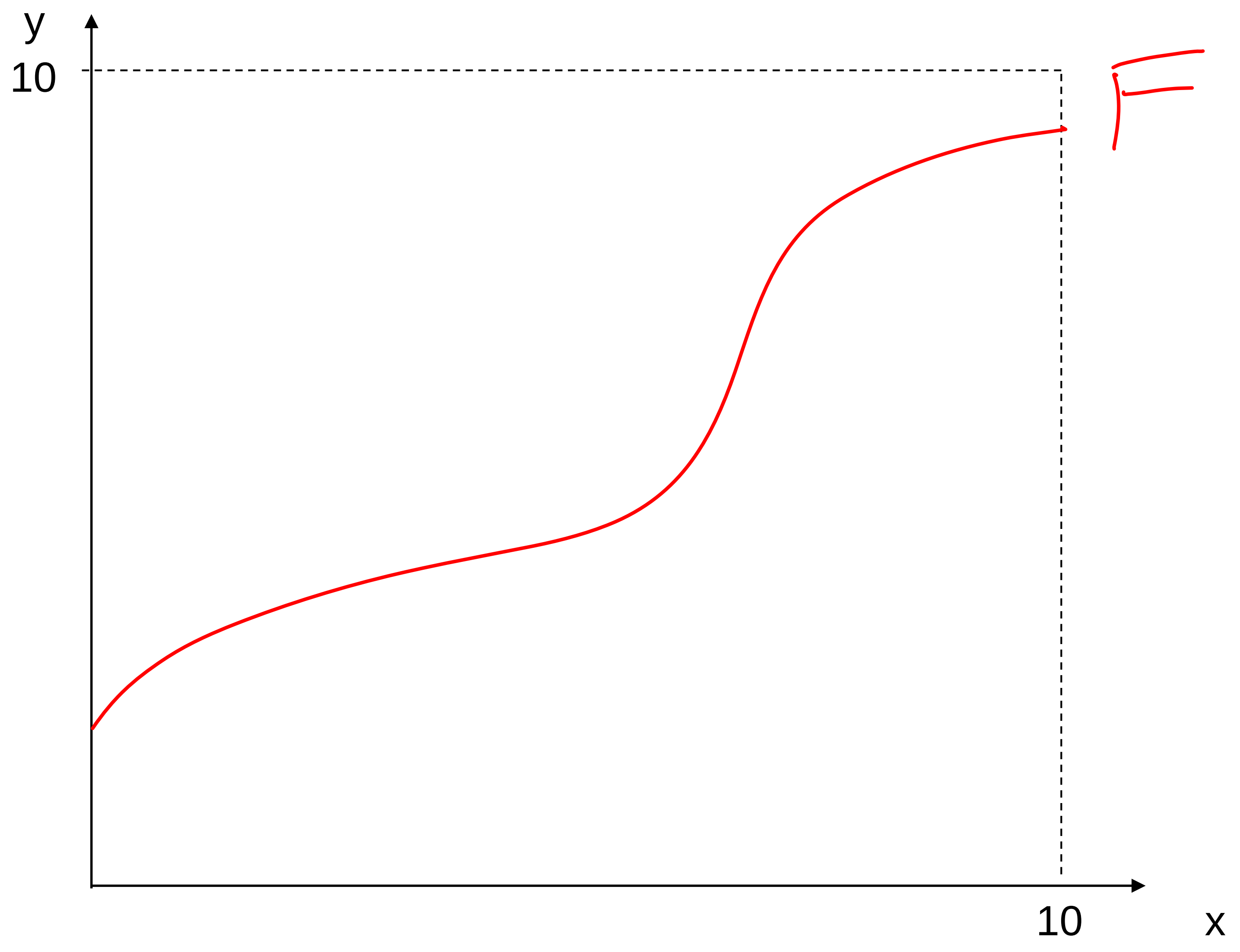
What about if we start at top?

- What if we start with $\top$: $F(\top)$, $F(F(\top))$, $F(F(F(\top)))$

What about if we start at top?

- What if we start with $\tilde{\top}$: $F(\tilde{\top})$, $F(F(\tilde{\top}))$, $F(F(F(\tilde{\top})))$
- We get the greatest fixed point
- Why do we prefer the least fixed point?
  - More precise

Graphically



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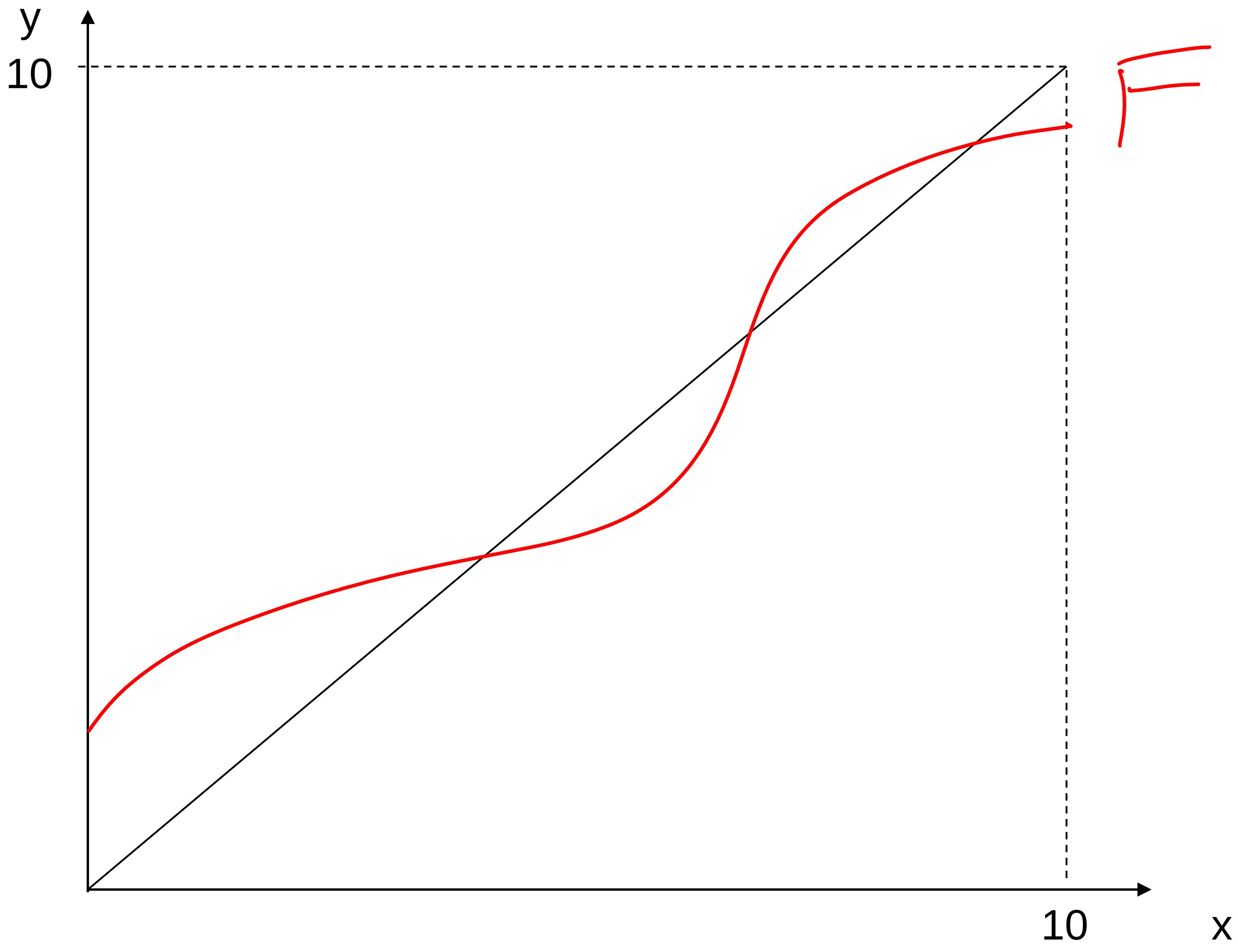

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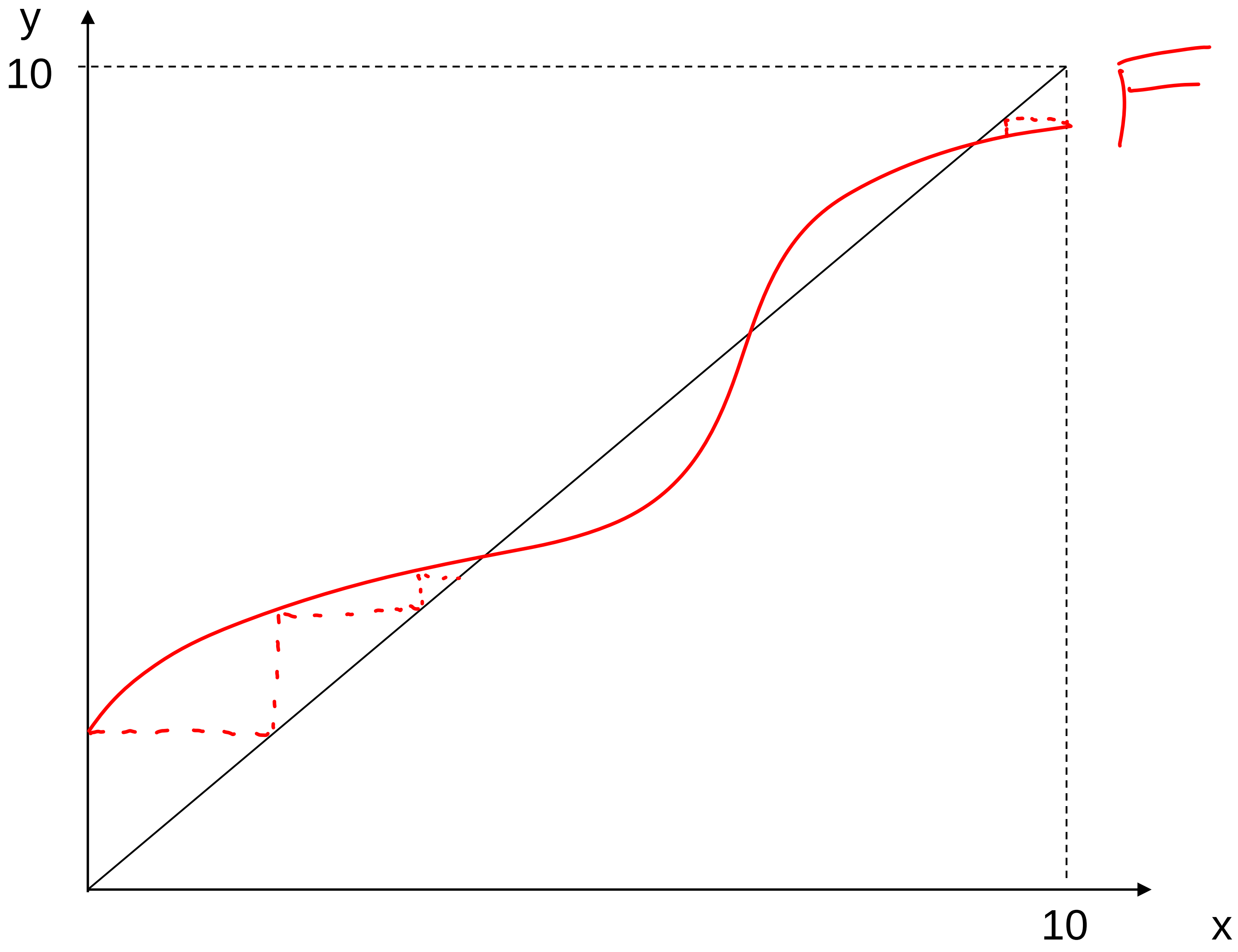

Graphically, another way