Background material

Relations

- A relation over a set S is a set $R \subseteq S \times S$
  - We write a R b for $(a,b) \in R$
- A relation R is:
  - reflexive iff

    $\forall\, a \in S\ .\ a\ R\ a$
  - transitive iff

    $\forall\, a \in S, b \in S, c \in S\ .\ a\ R\ b \wedge b\ R\ c \Rightarrow a\ R\ c$
  - symmetric iff

    $\forall\, a, b \in S\ .\ a\ R\ b \Rightarrow b\ R\ a$
  - anti-symmetric iff

    $\forall\, a, b, \in S\ .\ a\ R\ b \Rightarrow \neg(b\ R\ a)$

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  - anti-symmetric iff

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    $\forall\, a, b, \in S \,.\, a\, R\, b \wedge b\, R\, a \Rightarrow a = b$

Partial orders

- An equivalence class is a relation that is:

- A partial order is a relation that is:

Partial orders

- An equivalence class is a relation that is:
  – reflexive, transitive, symmetric
- A partial order is a relation that is:
  – reflexive, transitive, anti-symmetric
- A partially ordered set (a poset) is a pair $(S,\leq)$ of a set S and a partial order $\leq$ over the set
- Examples of posets: $(2^S, \subseteq)$, $(Z, \leq)$, $(Z, \text{divides})$ $a \bmod b = 0$

$\mathcal{P}(\{a,b\}) := \{\{a,b\}, \emptyset, \{a\}, \{b\}\}$

$2^{\{a,b\}}$ $\subseteq$

Lub and glb

- Given a poset $(S, \leq)$, and two elements $a \in S$ and $b \in S$, then the:
  - least upper bound (lub) is an element c such that
    $\underbrace{a \leq c,\ b \leq c,}$ and $\forall\, d \in S\ .\ (a \leq d \wedge b \leq d) \Rightarrow c \leq d$
  - greatest lower bound (glb) is an element c such that
    $c \leq a,\ c \leq b,$ and $\forall\, d \in S\ .\ (d \leq a \wedge d \leq b) \Rightarrow d \leq c$

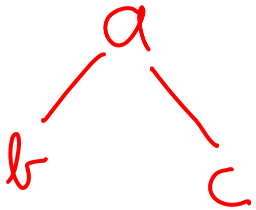
- Does a lub and glb always exists?

A. Yes (in this case justify your answer)

B. No (in this case come up with example where no lub or glb exists)

Lub and glb

- Given a poset (S, $\leq$), and two elements $a \in S$ and $b \in S$, then the:
  - least upper bound (lub) is an element c such that
    $a \leq c$, $b \leq c$, and $\forall\, d \in S\,.\, (a \leq d \wedge b \leq d) \Rightarrow c \leq d$
  - greatest lower bound (glb) is an element c such that
    $c \leq a$, $c \leq b$, and $\forall\, d \in S\,.\, (d \leq a \wedge d \leq b) \Rightarrow d \leq c$
- lub and glb don't always exists:



glb of b & c?

Lattices

- A lattice is a tuple $(S, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ such that:
  - $(S, \sqsubseteq)$ is a poset
  - $\forall a \in S . \bot \sqsubseteq a$
  - $\forall a \in S . a \sqsubseteq \top$
  - Every two elements from S have a lub and a glb
  - $\sqcup$ is the least upper bound operator, called a join
  - $\sqcap$ is the greatest lower bound operator, called a meet

$a \sqcap b$ $a \sqcup b$

Examples of lattices

- Powerset lattice

$S \triangleq \mathcal{P}(\{a,b,c\})$

$\sqsubseteq \;\triangleq\; \supseteq$

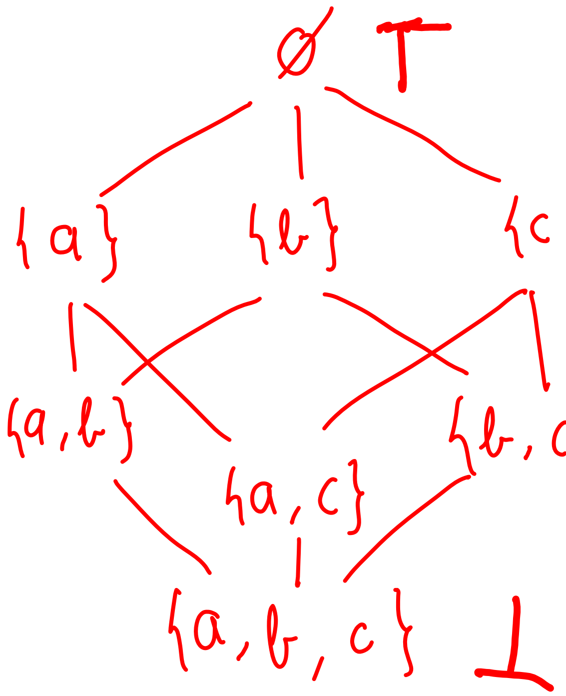
$\sqcup \;\triangleq\; \cap$

$\sqcap \;\triangleq\; \cup$

$\bot \sqsubseteq \top$

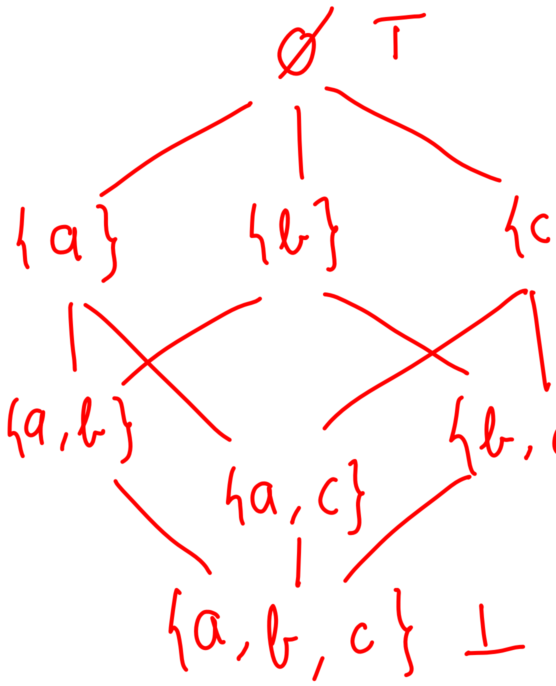
$\{a,b,c\} \sqsubseteq \emptyset$

$\subseteq$



Examples of lattices

- Powerset lattice



$S \triangleq 2^{\{a,b,c\}}$

$\sqsubseteq \triangleq \supseteq$

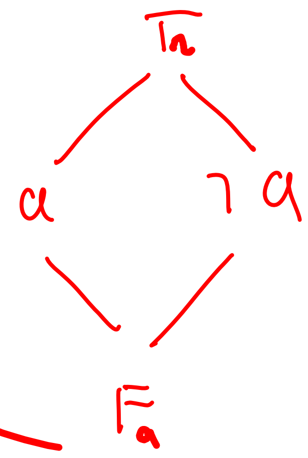
$\sqcup \triangleq \cap$

$\sqcap \triangleq \cup$

Examples of lattices

- Booleans expressions

one var



$\sqsubseteq \triangleq \Rightarrow$

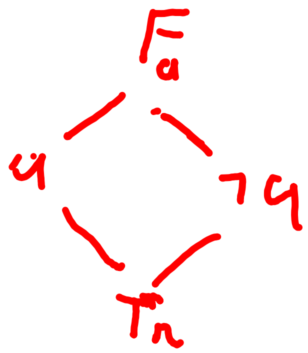
$\sqcup \triangleq \vee$

$\sqcap \triangleq \wedge$

$x \sqsubseteq y$ iff

A. $X \Rightarrow Y$

B $Y \Rightarrow X$



$\bot \sqsubseteq \top$

$F_a \sqsubseteq T_a$

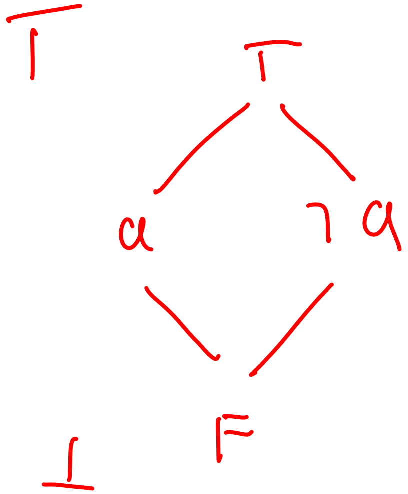
$\Rightarrow$

$\Leftarrow$

Examples of lattices

- Booleans expressions

one var



$\sqsubseteq \triangleq \Rightarrow$

$\sqcup \triangleq \vee$

$\sqcap \triangleq \wedge$

Examples of lattices

- Booleans expressions

two vars

T

⋮

F

Examples of lattices

- Booleans expressions

two vars

T

⋮ 16 expressions . . .

⋮

F

$2^{(2^n)}$

a	b	
0	0	$\square$
0	1	$\square$
1	0	$\square$
1	1	$\square$