Background material

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Relations

- A relation over a set S is a set $R \subseteq S \times S$
  - We write a R b for $(a,b) \in R$
- A relation R is:
  - reflexive iff
    $\forall a \in S \,.\, a\ R\ a$
  - transitive iff
    $\forall a \in S, b \in S, c \in S \,.\, a\ R\ b \wedge b\ R\ c \Rightarrow a\ R\ c$
  - symmetric iff
    $\forall a, b \in S \,.\, a\ R\ b \Rightarrow b\ R\ a$
  - anti-symmetric iff
    $\forall a, b, \in S \,.\, a\ R\ b \Rightarrow \neg(b\ R\ a)$

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Relations

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  - symmetric iff
    $\forall a, b \in S \,.\, a\ R\ b \Rightarrow b\ R\ a$
  - anti-symmetric iff
    $\cancel{\forall a, b, \in S \,.\, a\ R\ b \Rightarrow \neg(b\ R\ a)}$
    $\forall a, b, \in S \,.\, a\ R\ b \wedge b\ R\ a \Rightarrow a = b$

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Partial orders

- An equivalence class is a relation that is:
- A partial order is a relation that is:

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Partial orders

- An equivalence class is a relation that is:
  - reflexive, transitive, symmetric
- A partial order is a relation that is:
  - reflexive, transitive, anti-symmetric
- A partially ordered set (a poset) is a pair $(S,\leq)$ of a set S and a partial order $\leq$ over the set
- Examples of posets: $(2^S, \subseteq)$, $(Z, \leq)$, (Z, divides) $a \bmod b = 0$

$\mathcal{P}(\{a,b\}) = \{\{a,b\}, \emptyset, \{a\}, \{b\}\}$

$2^{\{a,b\}}$ $\subseteq$

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Lub and glb

- Given a poset $(S, \leq)$, and two elements $a \in S$ and $b \in S$, then the:
  - least upper bound (lub) is an element c such that
    $a \leq c, b \leq c$, and $\forall d \in S \,.\, (a \leq d \wedge b \leq d) \Rightarrow c \leq d$
  - greatest lower bound (glb) is an element c such that
    $c \leq a, c \leq b$, and $\forall d \in S \,.\, (d \leq a \wedge d \leq b) \Rightarrow d \leq c$
- Does a lub and glb always exists?

A. Yes (in this case justify your answer)

B. No (in this case come up with example where no lub or glb exists)

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Lub and glb

- Given a poset (S, $\leq$), and two elements $a \in S$ and $b \in S$, then the:
  - least upper bound (lub) is an element c such that $a \leq c$, $b \leq c$, and $\forall d \in S . (a \leq d \wedge b \leq d) \Rightarrow c \leq d$
  - greatest lower bound (glb) is an element c such that $c \leq a$, $c \leq b$, and $\forall d \in S . (d \leq a \wedge d \leq b) \Rightarrow d \leq c$
- lub and glb don't always exists:

glb of b & c?

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Lattices

- A lattice is a tuple (S, $\sqsubseteq$, $\bot$, $\top$, $\sqcup$, $\sqcap$) such that:
  - (S, $\sqsubseteq$) is a poset
  - $\forall a \in S . \bot \sqsubseteq a$
  - $\forall a \in S . a \sqsubseteq \top$
  - Every two elements from S have a lub and a glb
  - $\sqcup$ is the least upper bound operator, called a join
  - $\sqcap$ is the greatest lower bound operator, called a meet

$a \sqcap b$ $\quad a \sqcup b$

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Examples of lattices

- Powerset lattice

$S \triangleq \mathcal{P}(\{a,b,c\})$

$\sqsubseteq \triangleq \supseteq$

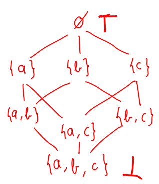
$\sqcup \triangleq \cap$

$\sqcap \triangleq \cup$

$\bot \sqsubseteq \top$

$\{a,b,c\} \sqsubseteq \emptyset$

$\subseteq$



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Examples of lattices

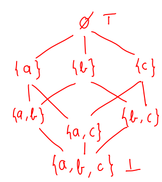
- Powerset lattice

$S \triangleq 2^{\{a,b,c\}}$

$\sqsubseteq \triangleq \supseteq$

$\sqcup \triangleq \cap$

$\sqcap \triangleq \cup$



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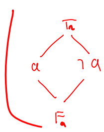
Examples of lattices

- Booleans expressions

one var



$\sqsubseteq \triangleq \Rightarrow$

$\sqcup \triangleq \vee$

$\sqcap \triangleq \wedge$

$x \sqsubseteq y$ iff

A. $x \Rightarrow y$

B. $y \Rightarrow x$

$\bot \sqsubseteq \top$

$F_a \sqsubseteq T_a$

$\Rightarrow$ $\Leftarrow$

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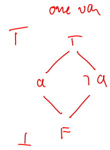
Examples of lattices

- Booleans expressions

one var



$\sqsubseteq \triangleq \Rightarrow$

$\sqcup \triangleq \vee$

$\sqcap \triangleq \wedge$

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Examples of lattices

- Booleans expressions

two ways

T

⋮

F

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Examples of lattices

- Booleans expressions

two ways

T

⋮ 16 expressions . . .

$2^{(2^n)}$

a	b	
0	0	□
0	1	□
1	0	□
1	1	□

F

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