


Formalization of DFA using lattices

Recall worklist algorithm

```
let m: map from edge to computed value at edge
let worklist: work list of nodes
```

```
for each edge e in CFG do
```

```
  m(e) :=  $\emptyset$  
```

```
for each node n do
```

```
  worklist.add(n)
```


```
while (worklist.empty.not) do
```

```
  let n := worklist.remove_any;
```

```
  let info_in := m(n.incoming_edges);
```

```
  let info_out := F(n, info_in);
```

```
  for i := 0 .. info_out.length do
```

```
    let new_info := m(n.outgoing_edges[i]) U 
      info_out[i];
```

```
    if (m(n.outgoing_edges[i])  $\neq$  new_info)
```

```
      m(n.outgoing_edges[i]) := new_info;
```

```
      worklist.add(n.outgoing_edges[i].dst);
```

Using lattices

- We formalize our domain with a powerset lattice
- What should be top and what should be bottom?

Using lattices

- We formalize our domain with a powerset lattice
- What should be top and what should be bottom?
- Does it matter?
 - It matters because, as we've seen, there is a notion of approximation, and this notion shows up in the lattice

Using lattices

- Unfortunately:
 - dataflow analysis community has picked one direction
 - abstract interpretation community has picked the other
- We will work with the abstract interpretation direction
- Bottom is the most precise (optimistic) answer, Top the most imprecise (conservative)

Direction of lattice

- Always safe to go up in the lattice
- Can always set the result to \top
- Hard to go down in the lattice
- Bottom will be the empty set in reaching defs

Worklist algorithm using lattices

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

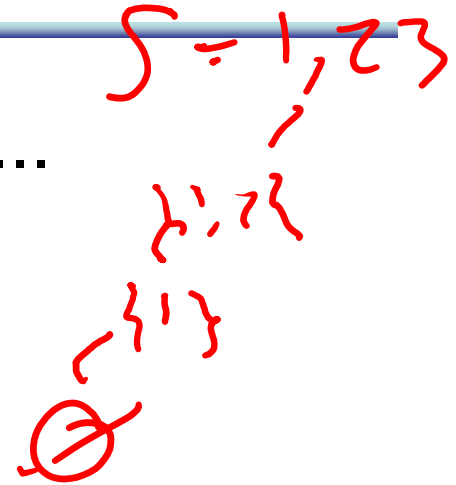
for each edge e in CFG do
  m(e) :=  $\perp$ 

for each node n do
  worklist.add(n)

while (worklist.empty.not) do
  let n := worklist.remove_any;
  let info_in := m(n.incoming_edges);
  let info_out := F(n, info_in);
  for i := 0 .. info_out.length do
    let new_info := m(n.outgoing_edges[i])  $\sqcup$ 
      info_out[i];
    if (m(n.outgoing_edges[i])  $\neq$  new_info)
      m(n.outgoing_edges[i]) := new_info;
      worklist.add(n.outgoing_edges[i].dst);
```

Termination of this algorithm?

- For reaching definitions, it terminates...
- Why?
 - lattice is finite
- Can we loosen this requirement?
 - Yes, we only require the lattice to have a finite height
- Height of a lattice: length of the longest ascending or descending chain
- Height of lattice $(2^S, \subseteq) = |S|$



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Termination

- Still, it's annoying to have to perform a join in the worklist algorithm

```
while (worklist.empty.not) do
  let n := worklist.remove_any;
  let info_in := m(n.incoming_edges);
  let info_out := F(n, info_in);
  for i := 0 .. info_out.length do
    let new_info := m(n.outgoing_edges[i])  $\sqcup$ 
      info_out[i];
    if (m(n.outgoing_edges[i])  $\neq$  new_info)
      m(n.outgoing_edges[i]) := new_info;
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```

- It would be nice to get rid of it, if there is a property of the flow functions that would allow us to do so

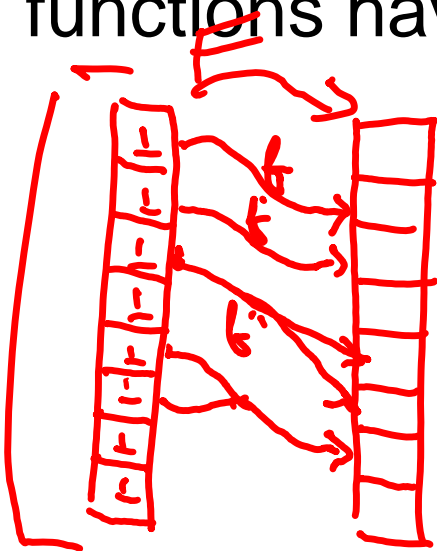
Even more formal

- To reason more formally about termination and precision, we re-express our worklist algorithm mathematically
- We will use fixed points to formalize our algorithm

Fixed points

$$\underline{f(x) = x^2}$$

- Recall, we are computing m , a map from edges to dataflow information
- Define a global flow function F as follows: F takes a map m as a parameter and returns a new map m' , in which individual local flow functions have been applied



$$m = F(m)$$



Fixed points

- We want to find a fixed point of F , that is to say a map m such that $m = F(m)$
- Approach to doing this?
- Define $\tilde{\perp}$, which is \perp lifted to be a map:
$$\tilde{\perp} = \lambda e. \perp$$
- Compute $F(\tilde{\perp})$, then $F(F(\tilde{\perp}))$, then $F(F(F(\tilde{\perp})))$, ... until the result doesn't change anymore

Fixed points

- Formally:

$$\text{Soln} = \bigsqcup_{i=0}^{\infty} F^i(\tilde{\perp})$$

- Outer join has same role here as in worklist algorithm: guarantee that results keep increasing
- BUT: if the sequence $F^i(\tilde{\perp})$ for $i = 0, 1, 2 \dots$ is increasing, we can get rid of the outer join!
- How? Require that F be monotonic:
 - $\forall a, b . a \sqsubseteq b \Rightarrow F(a) \sqsubseteq F(b)$

Fixed points

$$\begin{array}{ccc} \tilde{I} & \subseteq & F(\tilde{I}) \\ F(\tilde{I}) & \subseteq & FF(\tilde{I}) \\ FF(\tilde{I}) & \subseteq & FFF(\tilde{I}) \end{array}$$

Fixed points

$$\begin{aligned} \tilde{I} &\subseteq F(\tilde{I}) \\ F(\tilde{I}) &\subseteq F(F(\tilde{I})) \\ &\vdots \\ F^k(\tilde{I}) &\subseteq F^{k+1}(\tilde{I}) \\ F^{k+1}(\tilde{I}) &\subseteq F^{k+2}(\tilde{I}) \end{aligned}$$

Back to termination

- So if F is monotonic, we have what we want: finite height \Rightarrow termination, without the outer join
- Also, if the local flow functions are monotonic, then global flow function F is monotonic

Another benefit of monotonicity

- Suppose Marsians came to earth, and miraculously give you a fixed point of F , call it fp .
- Then:

$$\begin{aligned} \tilde{I} &\subseteq fp \\ F(\tilde{I}) &\subseteq fp \\ FF(\tilde{I}) &\subseteq fp \\ ofp &\subseteq fp \end{aligned}$$

Another benefit of monotonicity

- Suppose Marsians came to earth, and miraculously give you a fixed point of F , call it fp .
- Then:

$$\begin{aligned} \tilde{I} &\sqsubseteq fp \\ F(\tilde{I}) &\sqsubseteq F(fp) \\ F(\tilde{I}) &\sqsubseteq fp \\ F^2(\tilde{I}) &\sqsubseteq fp \\ &\vdots \\ 0fp &\sqsubseteq fp \end{aligned}$$

Another benefit of monotonicity

- We are computing the least fixed point...

Recap

- Let's do a recap of what we've seen so far
- Started with worklist algorithm for reaching definitions

Worklist algorithm for reaching defns

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
    m(e) :=  $\emptyset$ 

for each node n do
    worklist.add(n)

while (worklist.empty.not) do
    let n := worklist.remove_any;
    let info_in := m(n.incoming_edges);
    let info_out := F(n, info_in);
    for i := 0 .. info_out.length do
        let new_info := m(n.outgoing_edges[i]) U
            info_out[i];
        if (m(n.outgoing_edges[i])  $\neq$  new_info)
            m(n.outgoing_edges[i]) := new_info;
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```

Generalized algorithm using lattices

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
  m(e) :=  $\perp$ 

for each node n do
  worklist.add(n)

while (worklist.empty.not) do
  let n := worklist.remove_any;
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  for i := 0 .. info_out.length do
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      m(n.outgoing_edges[i]) := new_info;
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```

Next step: removed outer join

- Wanted to remove the outer join, while still providing termination guarantee
- To do this, we re-expressed our algorithm more formally
- We first defined a “global” flow function F , and then expressed our algorithm as a fixed point computation

Guarantees

- If F is monotonic, don't need outer join
- If F is monotonic and height of lattice is finite: iterative algorithm terminates
- If F is monotonic, the fixed point we find is the least fixed point.

What about if we start at top?

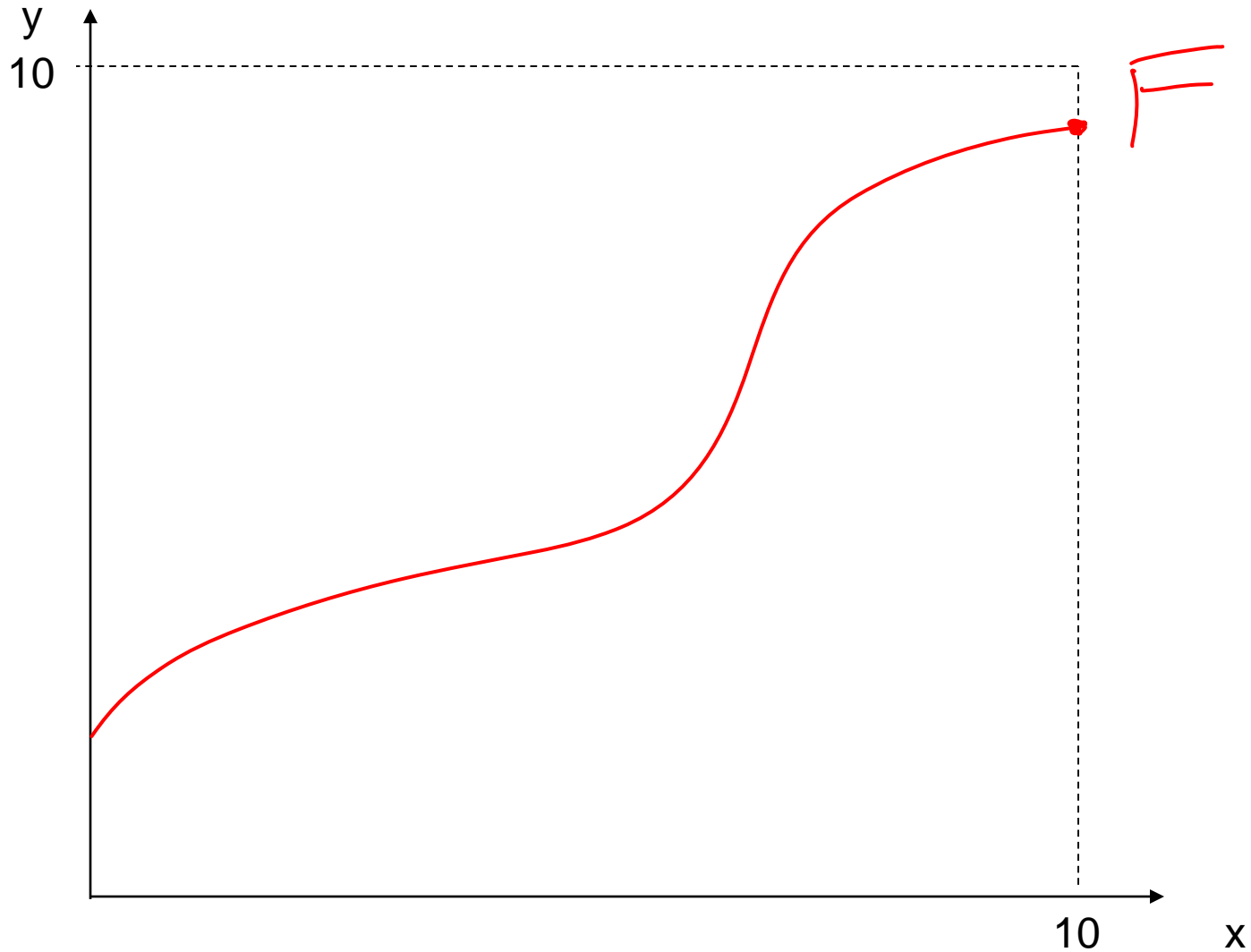
- What if we start with \tilde{T} : $F(\tilde{T})$, $F(F(\tilde{T}))$, $F(F(F(\tilde{T})))$

What about if we start at top?

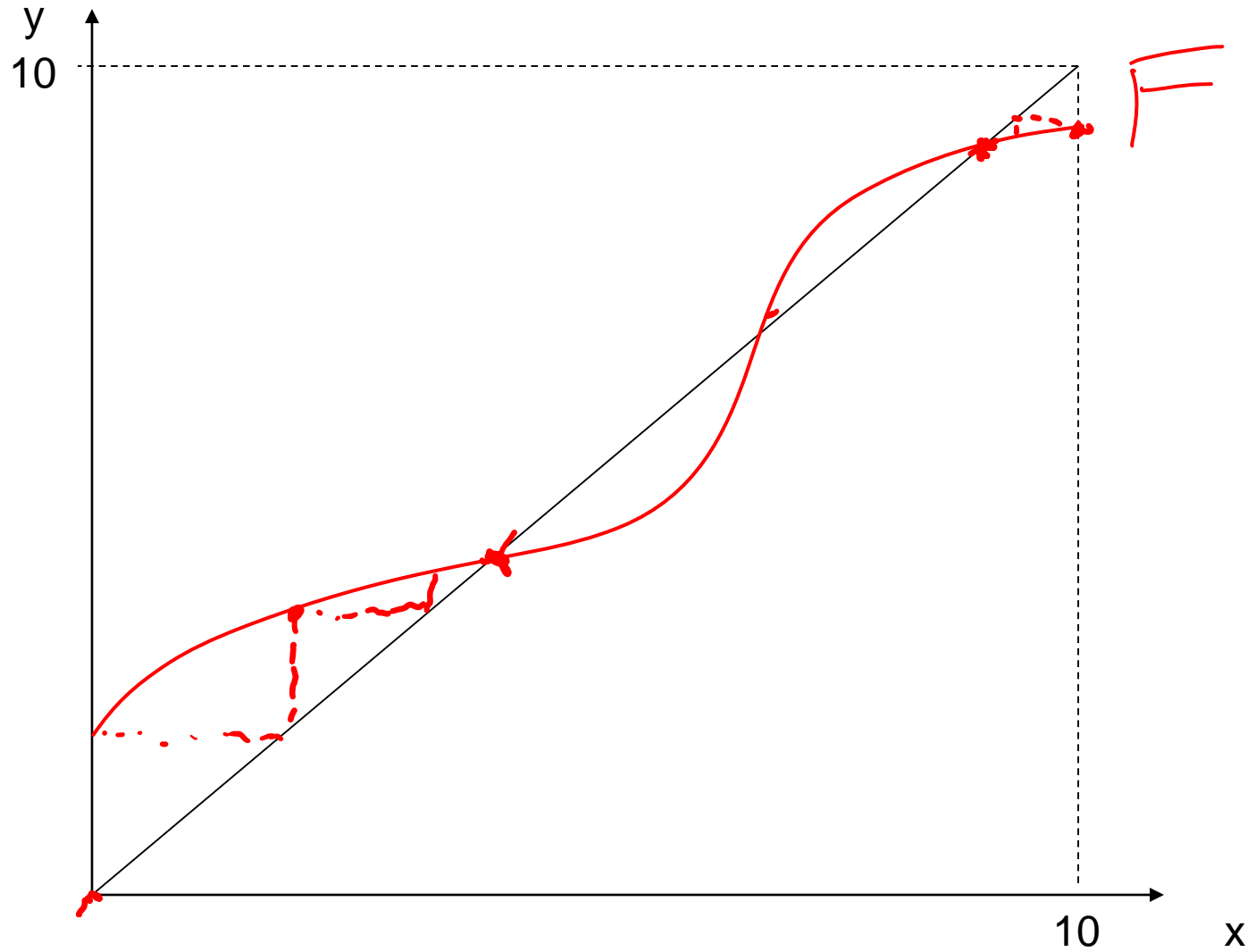
- What if we start with $\tilde{T}: F(\tilde{T}), F(F(\tilde{T})), F(F(F(\tilde{T})))$
- We get the greatest fixed point
- Why do we prefer the least fixed point?
 - More precise

Graphically

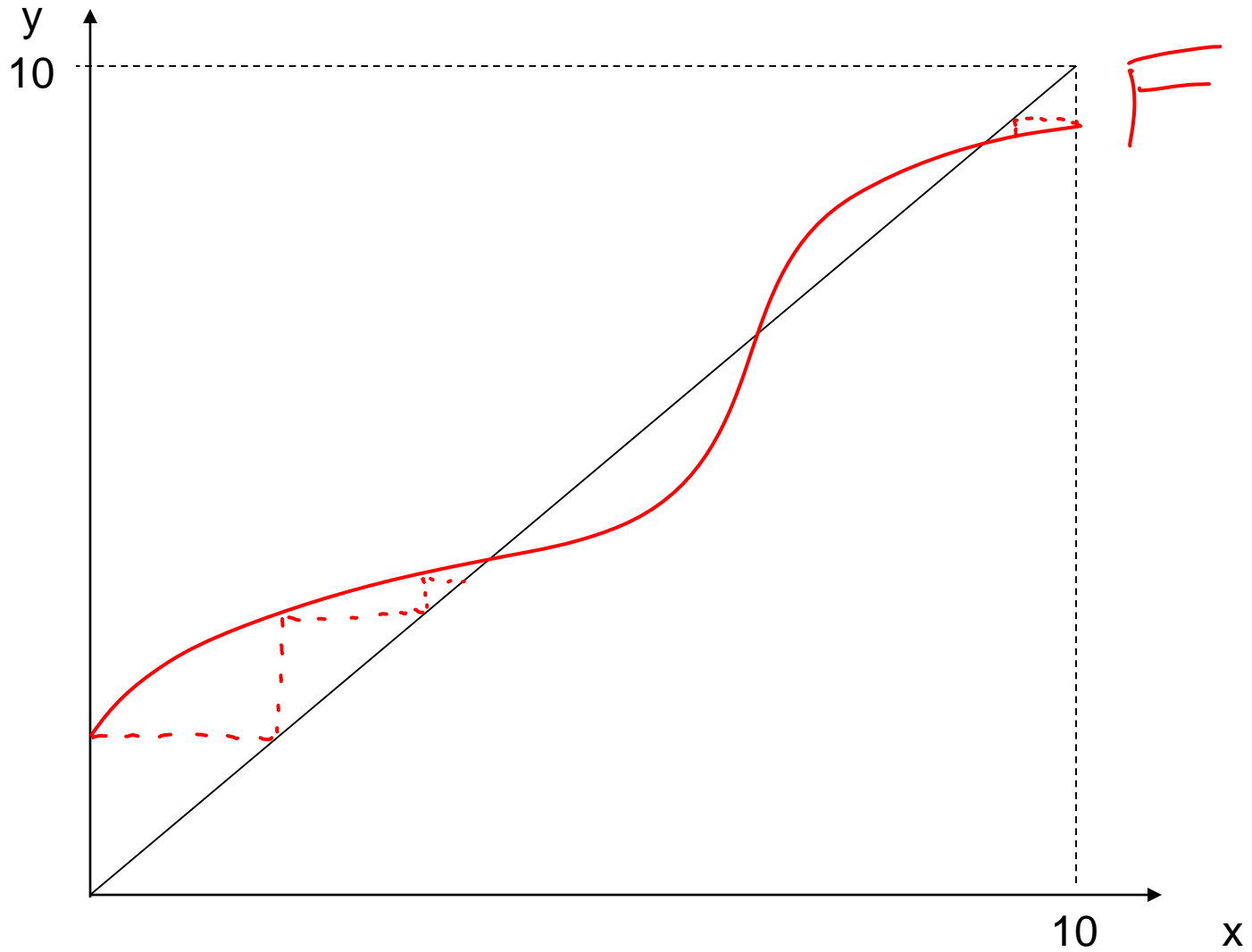
$$a \leq b \Rightarrow \underline{F(a)} \leq \underline{F(b)}$$



Graphically



Graphically



Graphically, another way
