

# Background material

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# Relations

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- A relation over a set  $S$  is a set  $R \subseteq S \times S$ 
  - We write  $a R b$  for  $(a,b) \in R$
- A relation  $R$  is:
  - reflexive iff
$$\forall a \in S . a R a$$
  - transitive iff
$$\forall a \in S, b \in S, c \in S . a R b \wedge b R c \Rightarrow a R c$$
  - symmetric iff
$$\forall a, b \in S . a R b \Rightarrow b R a$$
  - anti-symmetric iff
$$\forall a, b, \in S . a R b \Rightarrow \neg(b R a)$$

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$$\forall a, b, \in S . a R b \wedge b R a \Rightarrow a = b$$

# Partial orders

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- An equivalence class is a relation that is:
- A partial order is a relation that is:

# Partial orders

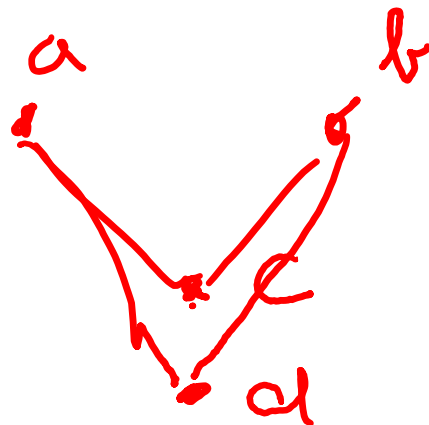
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- An equivalence class is a relation that is:
  - reflexive, transitive, symmetric
- A partial order is a relation that is:
  - reflexive, transitive, anti-symmetric
- A partially ordered set (a poset) is a pair  $(S, \leq)$  of a set  $S$  and a partial order  $\leq$  over the set
- Examples of posets:  $(2^S, \subseteq)$ ,  $(\mathbb{Z}, \leq)$ ,  $(\mathbb{Z}, \text{divides})$

# Lub and glb

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- Given a poset  $(S, \leq)$ , and two elements  $a \in S$  and  $b \in S$ , then the:
  - least upper bound (lub) is an element  $c$  such that  $a \leq c$ ,  $b \leq c$ , and  $\forall d \in S . (a \leq d \wedge b \leq d) \Rightarrow c \leq d$
  - greatest lower bound (glb) is an element  $c$  such that  $c \leq a$ ,  $c \leq b$ , and  $\forall d \in S . (d \leq a \wedge d \leq b) \Rightarrow d \leq c$



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- lub and glb don't always exist:

$a, b$

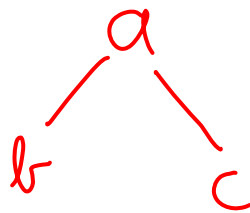
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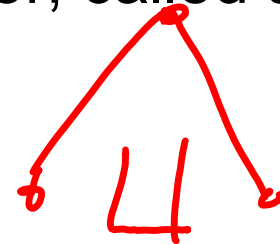
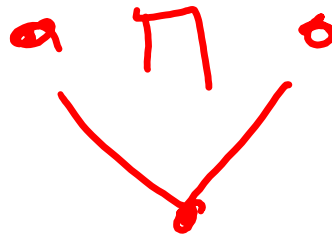
glb of  $b$  &  $c$ ?



# Lattices

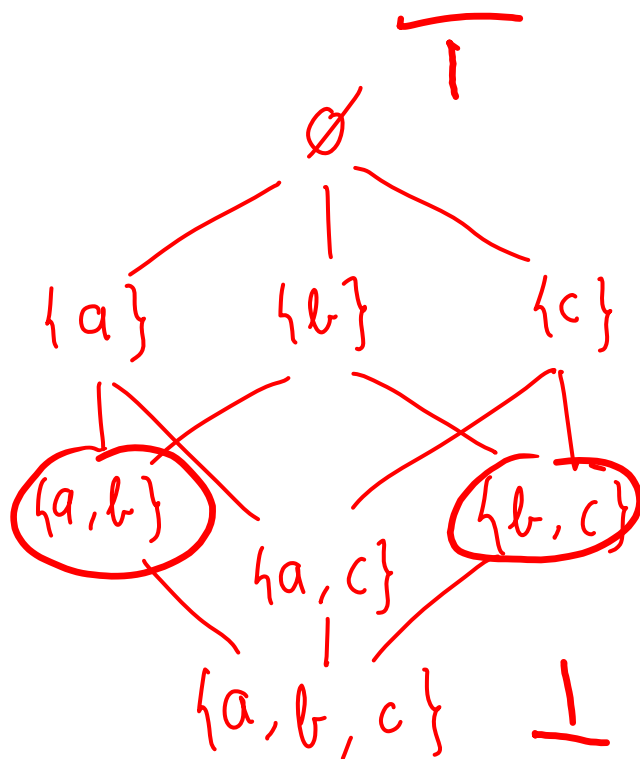
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- A lattice is a tuple  $(S, \sqsubseteq, \perp, \top, \sqcup, \sqcap)$  such that:
  - $(S, \sqsubseteq)$  is a poset
  - $\forall a \in S . \perp \sqsubseteq a$
  - $\forall a \in S . a \sqsubseteq \top$
  - Every two elements from  $S$  have a lub and a glb
  - $\sqcup$  is the least upper bound operator, called a join
  - $\sqcap$  is the greatest lower bound operator, called a meet



# Examples of lattices

- Powerset lattice



$$S \triangleq \mathcal{P}(\{a, b, c\})$$

$$\sqsubseteq \triangleq \supseteq$$

$$\sqcup \triangleq \cap$$

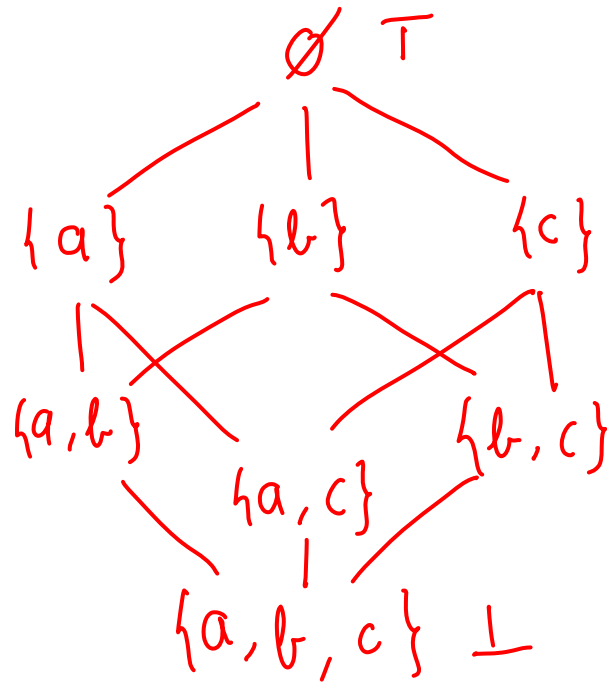
$$\sqcap \triangleq \cup$$

$$\perp \sqsubseteq \top$$

$$\{a, b, c\} \supseteq \emptyset$$

# Examples of lattices

- Powerset lattice



$$S \triangleq 2^{\{a, b, c\}}$$

$$\sqsubseteq \triangle \supseteq$$

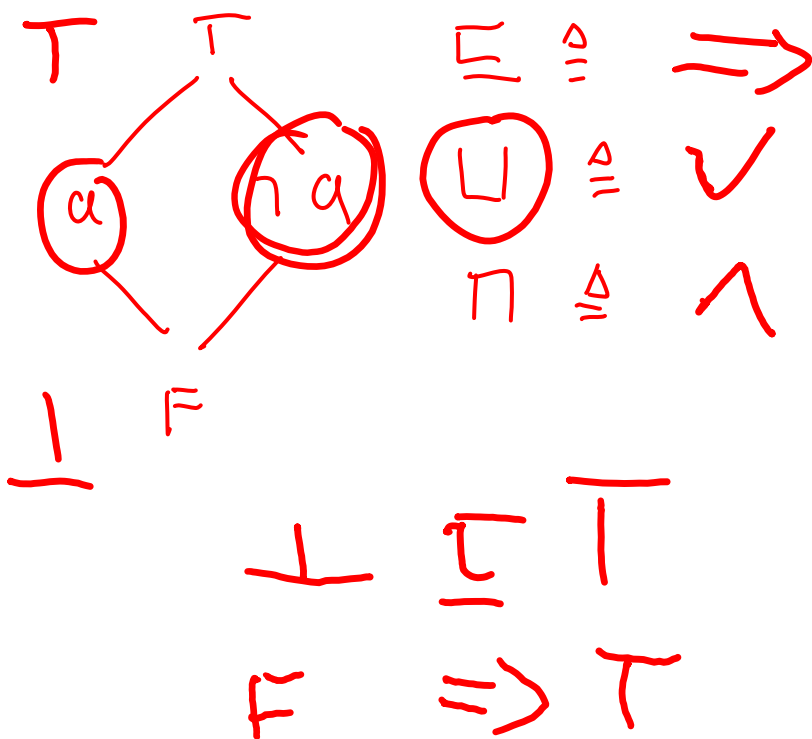
$$\sqcup \triangle \sqcap$$

$$\sqcap \triangle \sqcup$$

# Examples of lattices

- Booleans expressions

one var



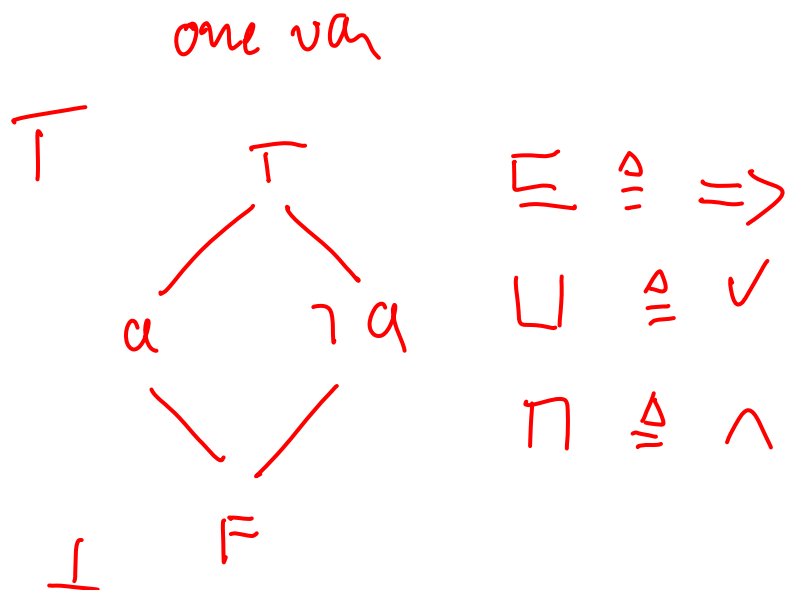
$a$	$\neg a$	$a \vee \neg a$	$a \wedge \neg a$
$0$	$1$	$1$	$0$
$1$	$0$	$1$	$0$

$\uparrow$     $\uparrow$     $\uparrow$     $\uparrow$

# Examples of lattices

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- Booleans expressions



# Examples of lattices

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- Booleans expressions

two way

T

⋮

F

# Examples of lattices

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- Booleans expressions

two vars

T

∴ 16 expressions ...

F

$2^{(2^2)}$

a	b	
0	0	□
0	1	□
1	0	□
1	1	□