Background material

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Relations

- A relation over a set S is a set $R \subseteq S \times S$
  - We write a R b for $(a,b) \in R$
- A relation R is:
  - reflexive iff
    $\forall a \in S . a R a$
  - transitive iff
    $\forall a \in S, b \in S, c \in S . a R b \wedge b R c \Rightarrow a R c$
  - symmetric iff
    $\forall a, b \in S . a R b \Rightarrow b R a$
  - anti-symmetric iff
    $\forall a, b, \in S . a R b \Rightarrow \neg(b R a)$

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  - symmetric iff
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  - anti-symmetric iff
    $\cancel{\forall a, b, \in S . a R b \Rightarrow \neg(b R a)}$
    $\forall a, b, \in S . a R b \wedge b R a \Rightarrow a = b$

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Partial orders

- An equivalence class is a relation that is:
- A partial order is a relation that is:

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Partial orders

- An equivalence class is a relation that is:
  - reflexive, transitive, symmetric
- A partial order is a relation that is:
  - reflexive, transitive, anti-symmetric
- A partially ordered set (a poset) is a pair $(S,\leq)$ of a set S and a partial order $\leq$ over the set
- Examples of posets: $(2^S, \subseteq)$, $(Z, \leq)$, (Z, divides)

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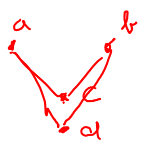
Lub and glb

- Given a poset $(S, \leq)$, and two elements $a \in S$ and $b \in S$, then the:
  - least upper bound (lub) is an element c such that $a \leq c$, $b \leq c$, and $\forall d \in S . (a \leq d \wedge b \leq d) \Rightarrow c \leq d$
  - greatest lower bound (glb) is an element c such that $c \leq a$, $c \leq b$, and $\forall d \in S . (d \leq a \wedge d \leq b) \Rightarrow d \leq c$



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Lub and glb

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- lub and glb don't always exists:

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Lub and glb

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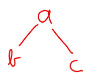
- lub and glb don't always exists:



glb of b & c?

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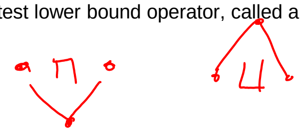
Lattices

- A lattice is a tuple (S, ⊑, ⊥, ⊤, ⊔, ⊓) such that:
  - (S, ⊑) is a poset
  - $\forall a \in S \,.\, \bot \sqsubseteq a$
  - $\forall a \in S \,.\, a \sqsubseteq \top$
  - Every two elements from S have a lub and a glb
  - ⊔ is the least upper bound operator, called a join
  - ⊓ is the greatest lower bound operator, called a meet



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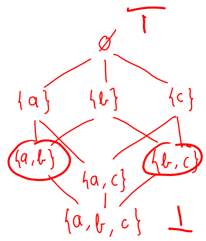
Examples of lattices

- Powerset lattice

$S \triangleq \wp(\{a,b,c\})$



$\sqsubseteq \triangleq \supseteq$

$\sqcup \triangleq \cap$

$\sqcap \triangleq \cup$

$\bot \sqsubseteq \top$

$\{a,b,c\} \supseteq \emptyset$

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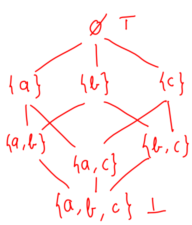
Examples of lattices

- Powerset lattice

$S \triangleq 2^{\{a,b,c\}}$



$\sqsubseteq \triangleq \supseteq$

$\sqcup \triangleq \cap$

$\sqcap \triangleq \cup$

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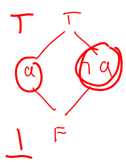
Examples of lattices

- Booleans expressions

one var



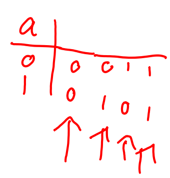
$\sqsubseteq \triangleq \Rightarrow$

$\sqcup \triangleq \vee$

$\sqcap \triangleq \wedge$

$\bot \sqsubseteq \top$

$F \Rightarrow T$



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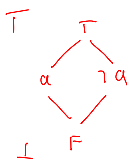
Examples of lattices

- Booleans expressions

one var



$\sqsubseteq \triangleq \Rightarrow$

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Examples of lattices

- Booleans expressions

two vars

T

⋮

F

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Examples of lattices

- Booleans expressions

two vars

T

⋮ 16 expressions...

$2^{(2^n)}$

a	b	
0	0	□
0	1	□
1	0	□
1	1	□

F

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