Formalization of DFA using lattices

Recall worklist algorithm

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
   m(e) := ∅ ⊥

for each node n do
   worklist.add(n)

while (worklist.empty.not) do
   let n := worklist.remove_any;
   let info_in := m(n.incoming_edges);
   let info_out := F(n, info_in);
   for i := 0 .. info_out.length do
      let new_info := m(n.outgoing_edges[i]) ∪ ⊔
                      info_out[i];
      if (m(n.outgoing_edges[i]) ≠ new_info])
         m(n.outgoing_edges[i]) := new_info;
         worklist.add(n.outgoing_edges[i].dst);
```

Using lattices

- We formalize our domain with a powerset lattice
- What should be top and what should be bottom?

Using lattices

- We formalize our domain with a powerset lattice
- What should be top and what should be bottom?
- Does it matter?
  - It matters because, as we've seen, there is a notion of approximation, and this notion shows up in the lattice

Using lattices

- Unfortunately:
  - dataflow analysis community has picked one direction
  - abstract interpretation community has picked the other
- We will work with the abstract interpretation direction
- Bottom is the most precise (optimistic) answer, Top the most imprecise (conservative)

Direction of lattice

- Always safe to go up in the lattice
- Can always set the result to $\top$
- Hard to go down in the lattice
- Bottom will be the empty set in reaching defs

Worklist algorithm using lattices

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let worklist: work list of nodes

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while (worklist.empty.not) do
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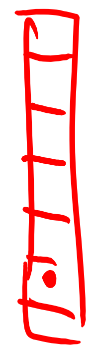

Termination of this algorithm?

- For reaching definitions, it terminates...
- Why?
  - lattice is finite
- Can we loosen this requirement?
  - Yes, we only require the lattice to have a finite height
- Height of a lattice: length of the longest ascending or descending chain
- Height of lattice (2^S, $\subseteq$) = $|S|$

$\mathcal{P}(S)$

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- Height of a lattice: length of the longest ascending or descending chain
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Termination

- Still, it’s annoying to have to perform a join in the worklist algorithm

```
while (worklist.empty.not) do
   let n := worklist.remove_any;
   let info_in := m(n.incoming_edges);
   let info_out := F(n, info_in);
   for i := 0 .. info_out.length do
      let new_info := m(n.outgoing_edges[i]) ⊔
                      info_out[i];
      if (m(n.outgoing_edges[i]) ≠ new_info])
         m(n.outgoing_edges[i]) := new_info;
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```

- It would be nice to get rid of it, if there is a property of the flow functions that would allow us to do so

Even more formal

- To reason more formally about termination and precision, we re-express our worklist algorithm mathematically

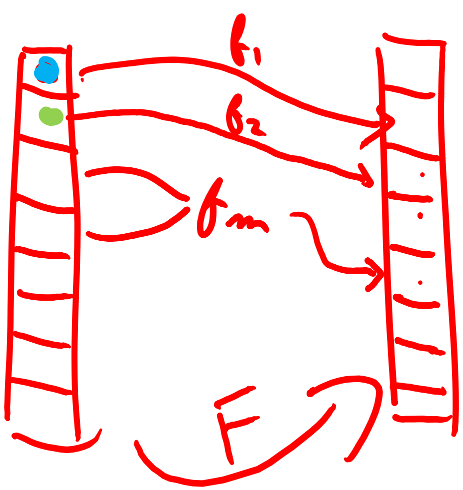
- We will use fixed points to formalize our algorithm

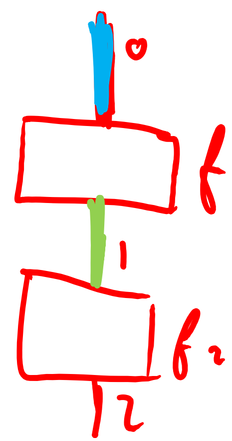
Fixed points

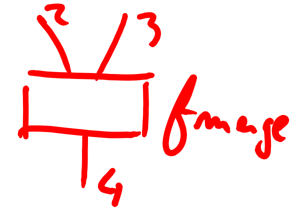
$g(x) = x$

- Recall, we are computing m, a map from edges to dataflow information

- Define a global flow function F as follows: F takes a map m as a parameter and returns a new map m', in which individual local flow functions have been applied







Fixed points

- We want to find a fixed point of F, that is to say a map m such that m = F(m)
- Approach to doing this?
- Define $\tilde{\perp}$, which is $\perp$ lifted to be a map:

  $\tilde{\perp} = \lambda$ e. $\perp$

- Compute F($\tilde{\perp}$), then F(F($\tilde{\perp}$)), then F(F(F($\tilde{\perp}$))), ... until the result doesn't change anymore

Fixed points

- Formally:

$$\text{Soln} = \bigsqcup_{i=0}^{\infty} F^i(\tilde{\perp})$$

- Outer join has same role here as in worklist algorithm: guarantee that results keep increasing
- BUT: if the sequence $F^i(\tilde{\perp})$ for i = 0, 1, 2 ... is increasing, we can get rid of the outer join!
- How? Require that F be monotonic:
  - $\forall$ a, b . a $\sqsubseteq$ b $\Rightarrow$ F(a) $\sqsubseteq$ F(b)

$a \qquad F(a) \qquad F(a)$

Fixed points

$$\perp \sqsubseteq F(\perp)$$

$$F(\perp) \sqsubseteq FF(\perp)$$

$$FF(\perp) \sqsubseteq FFF(\perp)$$

Fixed points

$$\tilde{\bot} \;\tilde{\sqsubseteq}\; F(\tilde{\bot})$$

$$F(\tilde{\bot}) \sqsubseteq F(F(\tilde{\bot}))$$

$$F^{k}(\tilde{\bot}) \sqsubseteq F^{k+1}(\tilde{\bot})$$

$$F^{k+1}(\tilde{\bot}) \sqsubseteq F^{k+2}(\tilde{\bot})$$

Back to termination

- So if F is monotonic, we have what we want: finite height $\Rightarrow$ termination, without the outer join
- Also, if the local flow functions are monotonic, then global flow function F is monotonic

Another benefit of monotonicity

- Suppose Marsians came to earth, and miraculously give you a fixed point of F, call it fp.
- Then:

$$\perp \sqsubseteq fp$$

$$F(\perp) \sqsubseteq fp$$

$$FF(\perp) \sqsubseteq fp$$

$$\underbrace{\text{s}fp}_{,} \sqsubseteq fp$$

Another benefit of monotonicity

- Suppose Marsians came to earth, and miraculously give you a fixed point of F, call it fp.
- Then:

$$\tilde{\bot} \sqsubseteq fp$$

$$F(\tilde{\bot}) \sqsubseteq F(fp)$$

$$F(\tilde{\bot}) \sqsubseteq fp$$

$$F^2(\tilde{\bot}) \sqsubseteq fp$$

$$\vdots$$

$$ofp \sqsubseteq fp$$

Another benefit of monotonicity

- We are computing the least fixed point...

Recap

- Let's do a recap of what we've seen so far
- Started with worklist algorithm for reaching definitions

Worklist algorithm for reaching defns

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
   m(e) := ∅

for each node n do
   worklist.add(n)

while (worklist.empty.not) do
   let n := worklist.remove_any;
   let info_in := m(n.incoming_edges);
   let info_out := F(n, info_in);
   for i := 0 .. info_out.length do
      let new_info := m(n.outgoing_edges[i]) ∪
                      info_out[i];
      if (m(n.outgoing_edges[i]) ≠ new_info])
         m(n.outgoing_edges[i]) := new_info;
         worklist.add(n.outgoing_edges[i].dst);
```

Generalized algorithm using lattices

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
   m(e) := ⊥

for each node n do
   worklist.add(n)

while (worklist.empty.not) do
   let n := worklist.remove_any;
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   let info_out := F(n, info_in);
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```

Next step: removed outer join

- Wanted to remove the outer join, while still providing termination guarantee

- To do this, we re-expressed our algorithm more formally

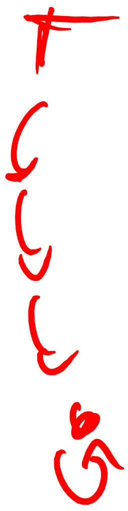
- We first defined a “global” flow function F, and then expressed our algorithm as a fixed point computation

Guarantees

- If F is monotonic, don’t need outer join
- If F is monotonic and height of lattice is finite: iterative algorithm terminates
- If F is monotonic, the fixed point we find is the least fixed point.

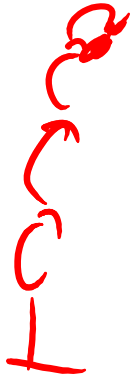
What about if we start at top?

- What if we start with $\tilde{\top}$: $F(\tilde{\top})$, $F(F(\tilde{\top}))$, $F(F(F(\tilde{\top})))$



$F(\top) \subseteq \top$

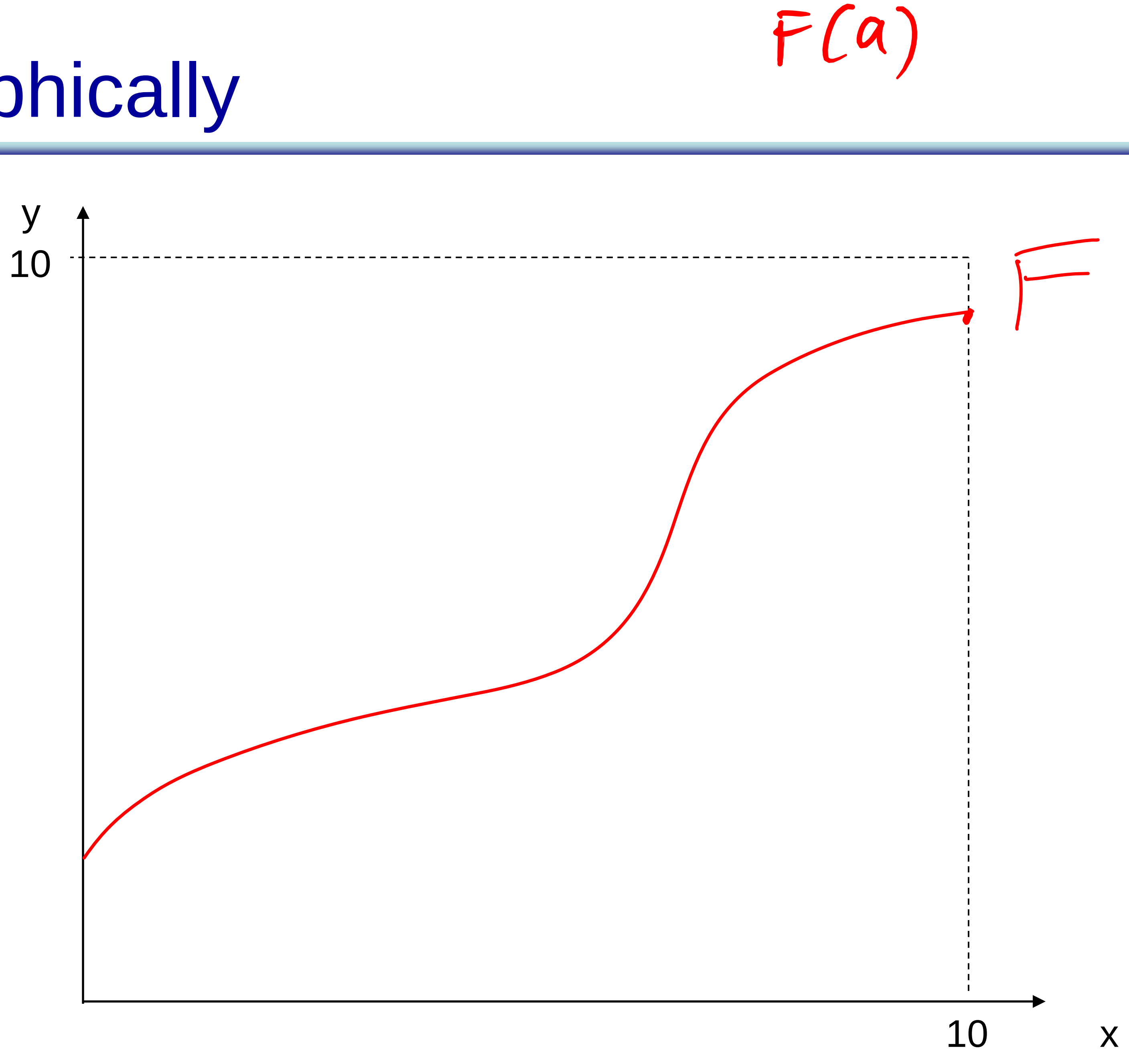
$FF(\top) \subseteq F(\top)$



What about if we start at top?

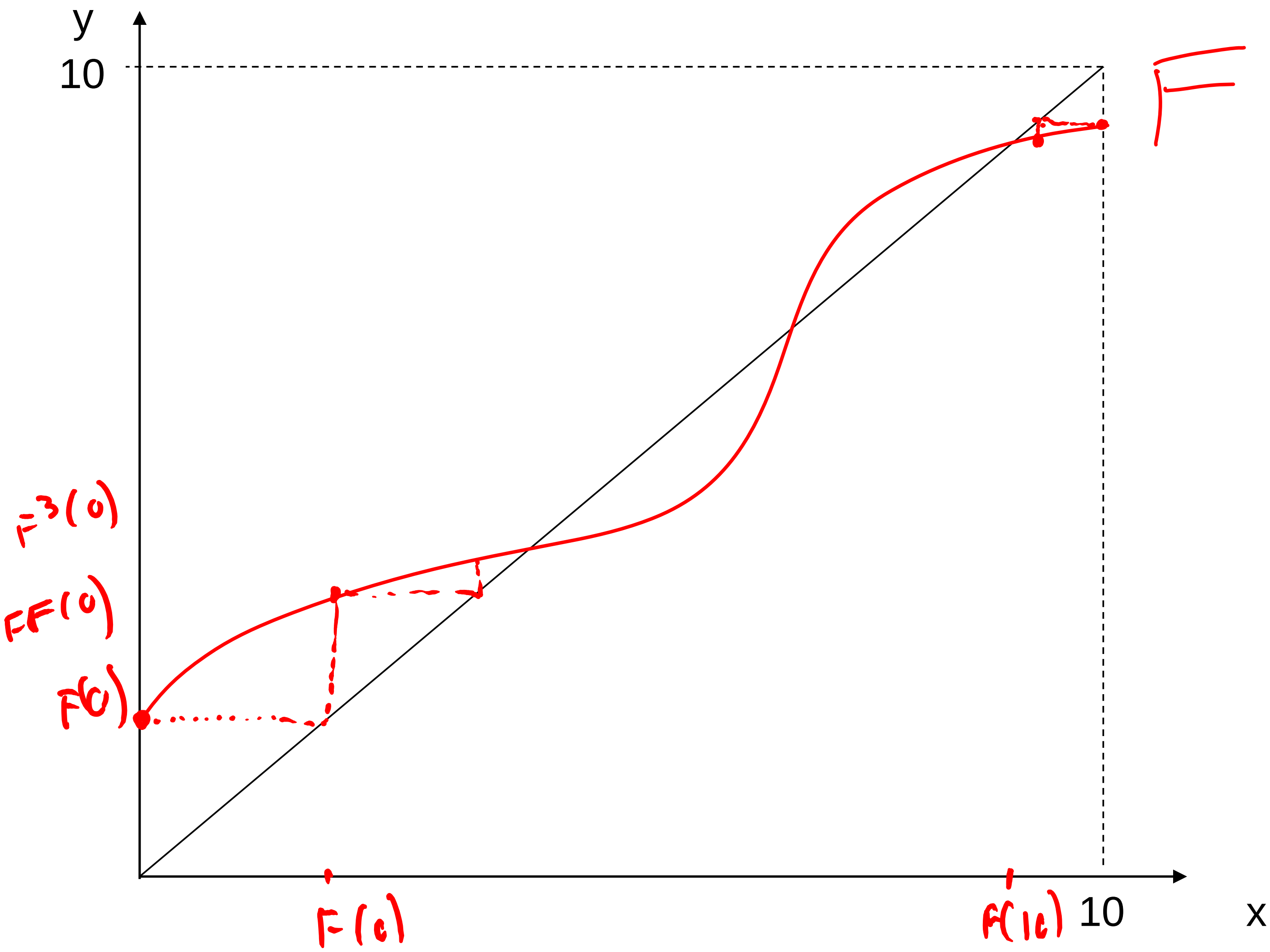
- What if we start with $\top$: $F(\top)$, $F(F(\top))$, $F(F(F(\top)))$
- We get the greatest fixed point
- Why do we prefer the least fixed point?
  - More precise

Graphically

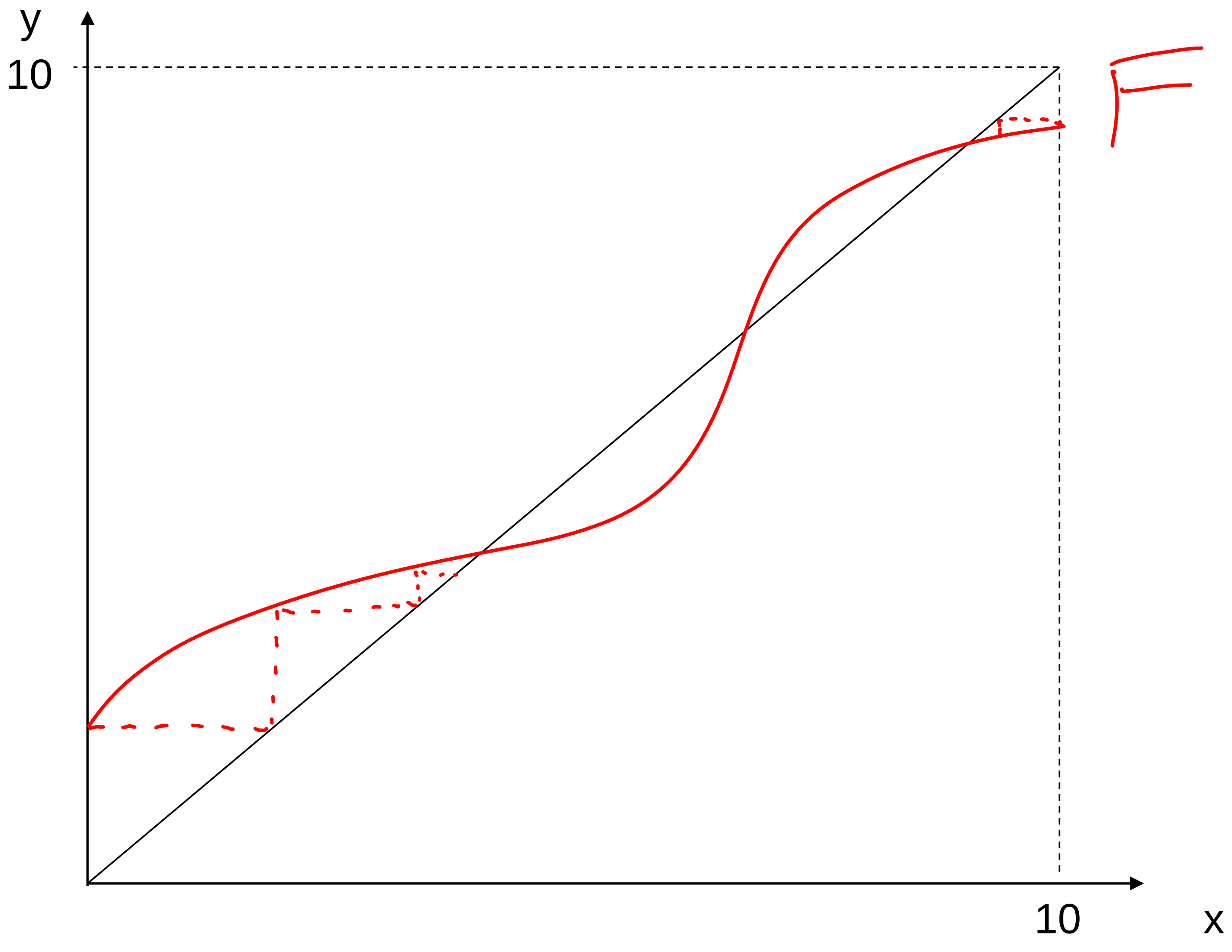


Graphically

$F(10) < 10$



Graphically



Graphically, another way