Background material

Relations

- A relation over a set S is a set $R \subseteq S \times S$
  - We write a R b for $(a,b) \in R$
- A relation R is:
  - reflexive iff

    $\forall a \in S \,.\, a \; R \; a$
  - transitive iff

    $\forall a \in S, b \in S, c \in S \,.\, a \; R \; b \wedge b \; R \; c \Rightarrow a \; R \; c$
  - symmetric iff

    $\forall a, b \in S \,.\, a \; R \; b \Rightarrow b \; R \; a$
  - anti-symmetric iff

    $\forall a, b, \in S \,.\, a \; R \; b \Rightarrow \neg(b \; R \; a)$

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    $\cancel{\forall a, b, \in S . a R b \Rightarrow \neg(b R a)}$

    $\forall a, b, \in S . a R b \wedge b R a \Rightarrow a = b$

Partial orders

- An equivalence class is a relation that is:

- A partial order is a relation that is:

$<=$

$>=$

$\subseteq$

Partial orders

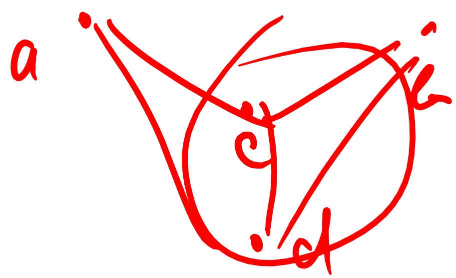
- An equivalence class is a relation that is:
  - reflexive, transitive, symmetric
- A partial order is a relation that is:
  - reflexive, transitive, anti-symmetric
- A partially ordered set (a poset) is a pair $(S, \leq)$ of a set S and a partial order $\leq$ over the set
- Examples of posets: $(2^S, \subseteq)$, $(Z, \leq)$, (Z, divides)

$\{5\}$ $\{7\}$

$\subseteq$

Lub and glb

- Given a poset $(S, \leq)$, and two elements $a \in S$ and $b \in S$, then the:
  - least upper bound (lub) is an element c such that $a \leq c$, $b \leq c$, and $\forall\, d \in S\ .\ (a \leq d \wedge b \leq d) \Rightarrow c \leq d$
  - greatest lower bound (glb) is an element c such that $c \leq a$, $c \leq b$, and $\forall\, d \in S\ .\ (d \leq a \wedge d \leq b) \Rightarrow d \leq c$



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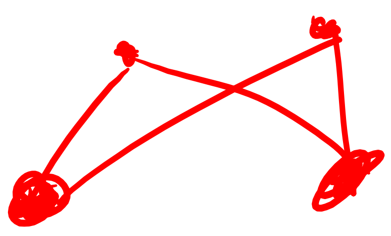
- lub and glb don’t always exists:



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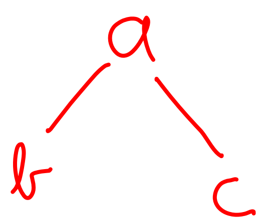


glb of b & c ?

Lattices

- A lattice is a tuple $(S, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ such that:
  - $(S, \sqsubseteq)$ is a poset
  - $\forall a \in S . \bot \sqsubseteq a$
  - $\forall a \in S . a \sqsubseteq \top$
  - Every two elements from S have a lub and a glb
  - $\sqcup$ is the least upper bound operator, called a join
  - $\sqcap$ is the greatest lower bound operator, called a meet

$a \sqcap b$

Examples of lattices

- Powerset lattice

$S \triangleq \mathcal{P}(\{a,b,c\})$

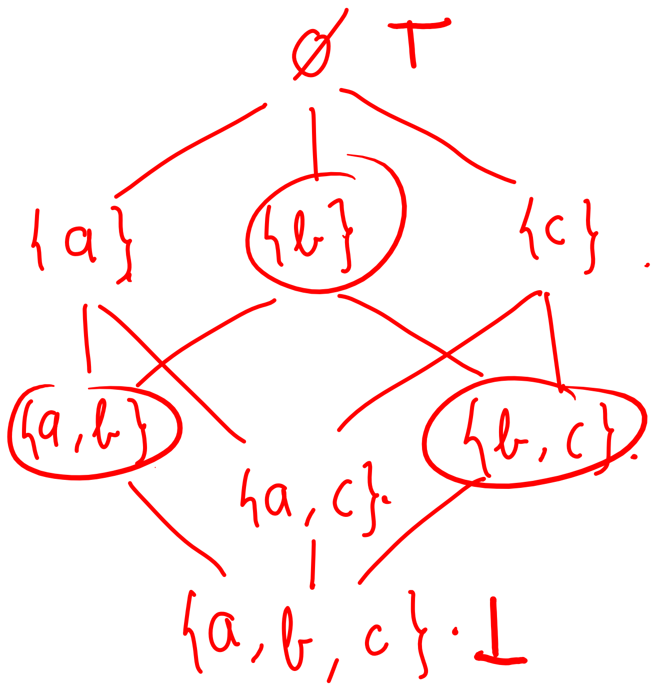
$\sqsubseteq \; \triangleq \; \supseteq$

$\sqcup \; \triangleq \; \cap$

$\sqcap \; \triangleq \; \cup$

$\bot \sqsubseteq \top$

$\{a,b,c\} \supseteq \emptyset$



Examples of lattices

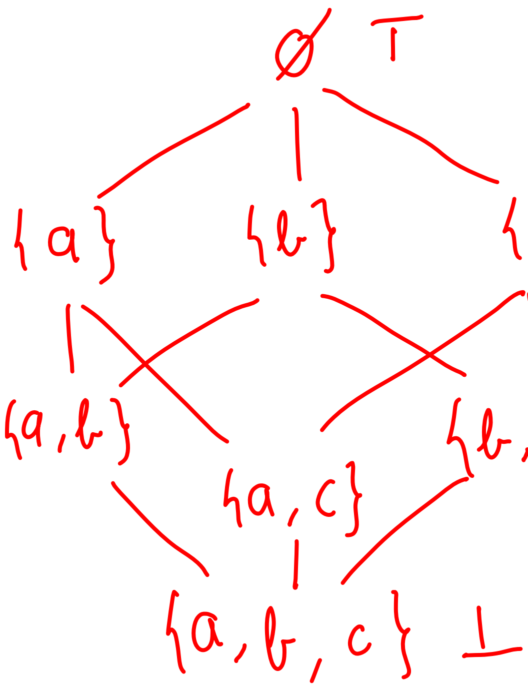
- Powerset lattice

$S \triangleq 2^{\{a,b,c\}}$

$\sqsubseteq \triangleq \supseteq$

$\sqcup \triangleq \cap$

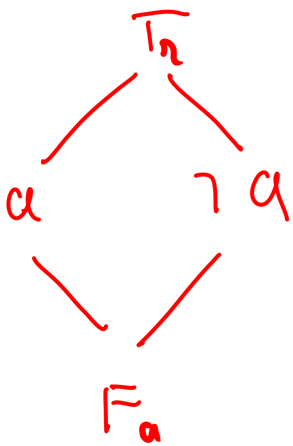
$\sqcap \triangleq \cup$



Examples of lattices

- Booleans expressions

one var



$\sqsubseteq \triangleq \rightarrow$

$\sqcup^{J} \triangleq \vee$

$\sqcap^{m} \triangleq \wedge$

Examples of lattices

- Booleans expressions

one var

$$\begin{tikzcd}[column sep=small]
\top & & T \arrow[dl, dash] \arrow[dr, dash] & \\
& a \arrow[dr, dash] & & \neg a \arrow[dl, dash] \\
\bot & & F &
\end{tikzcd}$$

$\sqsubseteq \triangleq \Rightarrow$

$\sqcup \triangleq \vee$

$\sqcap \triangleq \wedge$

Examples of lattices

- Booleans expressions

two vars

T

⋮

F

Examples of lattices

- Booleans expressions

two vars

T

⋮ 16 expressions ...

⋮

F

$$2^{(2^n)}$$

a	b	
0	0	□
0	1	□
1	0	□
1	1	□