Background material

Relations

- A relation over a set S is a set R $\subseteq$ S $\times$ S
  - We write a R b for (a,b) $\in$ R
- A relation R is:
  - reflexive iff
    $\forall$ a $\in$ S . a R a
  - transitive iff
    $\forall$ a $\in$ S, b $\in$ S, c $\in$ S . a R b $\wedge$ b R c $\Rightarrow$ a R c
  - symmetric iff
    $\forall$ a, b $\in$ S . a R b $\Rightarrow$ b R a
  - anti-symmetric iff
    $\forall$ a, b, $\in$ S . a R b $\Rightarrow$ $\neg$(b R a)

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    $\forall$ a, b, $\in$ S . a R b $\wedge$ b R a $\Rightarrow$ a = b

Partial orders

- An equivalence class is a relation that is:
- A partial order is a relation that is:

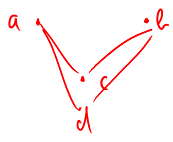
Partial orders

- An equivalence class is a relation that is:
  - reflexive, transitive, symmetric =
- A partial order is a relation that is:
  - reflexive, transitive, anti-symmetric
- A partially ordered set (a poset) is a pair (S,$\leq$) of a set S and a partial order $\leq$ over the set
- Examples of posets: (2^S, $\subseteq$), (Z, $\leq$), (Z, divides)

$\mathcal{P}(S)$

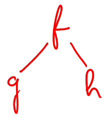
Lub and glb

- Given a poset (S, $\leq$), and two elements a $\in$ S and b $\in$ S, then the:
  - least upper bound (lub) is an element c such that a $\leq$ c, b $\leq$ c, and $\forall$ d $\in$ S . (a $\leq$ d $\wedge$ b $\leq$ d) $\Rightarrow$ c $\leq$ d
  - greatest lower bound (glb) is an element c such that c $\leq$ a, c $\leq$ b, and $\forall$ d $\in$ S . (d $\leq$ a $\wedge$ d $\leq$ b) $\Rightarrow$ d $\leq$ c



Lub and glb

- Given a poset $(S, \leq)$, and two elements $a \in S$ and $b \in S$, then the:
  - least upper bound (lub) is an element c such that $a \leq c$, $b \leq c$, and $\forall\, d \in S \,.\, (a \leq d \wedge b \leq d) \Rightarrow c \leq d$
  - greatest lower bound (glb) is an element c such that $c \leq a$, $c \leq b$, and $\forall\, d \in S \,.\, (d \leq a \wedge d \leq b) \Rightarrow d \leq c$
- lub and glb don't always exists:



Lub and glb

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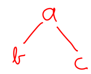


glb of b & c?

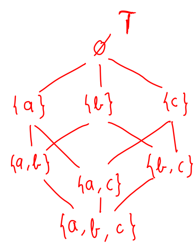
Lattices

- A lattice is a tuple $(S, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ such that:
  - $(S, \sqsubseteq)$ is a poset
  - $\forall\, a \in S \,.\, \bot \sqsubseteq a$
  - $\forall\, a \in S \,.\, a \sqsubseteq \top$
  - Every two elements from S have a lub and a glb
  - $\sqcup$ is the least upper bound operator, called a join
  - $\sqcap$ is the greatest lower bound operator, called a meet

$\sqcup$ $\sqcap$

Examples of lattices

- Powerset lattice



$S \triangleq \mathcal{P}(\{a,b,c\})$

$\sqsubseteq \triangleq \supseteq$

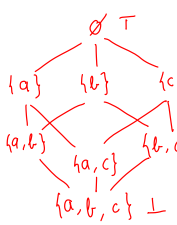
$\sqcup \triangleq \cap$

$\sqcap \triangleq \cup$

Examples of lattices

- Powerset lattice

$S \triangleq 2^{\{a,b,c\}}$



$\sqsubseteq \triangleq \supseteq$

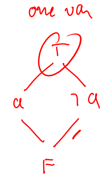
$\sqcup \triangleq \cap$

$\sqcap \triangleq \cup$

Examples of lattices

- Booleans expressions

one var



$\sqsubseteq \triangleq \Rightarrow$

$\sqcup \triangleq \vee$

$\sqcap \triangleq \wedge$

Examples of lattices

- Booleans expressions

one way

$\top$

$$T \quad a \quad \neg a \quad F$$

$\bot$

$\sqsubseteq \triangleq \Rightarrow$

$\sqcup \triangleq \vee$

$\sqcap \triangleq \wedge$

Examples of lattices

- Booleans expressions

two ways

T

⋮

F

Examples of lattices

- Booleans expressions

two ways

T

⋮ 16 expressions...

F

$2^{(2^n)}$

a	b	
0	0	$\square$
0	1	$\square$
1	0	$\square$
1	1	$\square$